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Abstract. The relativistic two-body system in (1 + 1)-dimensional quantum electrodynamics is studied. It is proved that the eigenvalue problem for the two-body Hamiltonian without the self-interaction terms reduces to the problem of solving a one-dimensional stationary Schrödinger-type equation with an energy-dependent effective potential which includes the δ-functional and inverted oscillator parts. The conditions determining the metastable energy spectrum are derived, and the energies and widths of the metastable levels are estimated in the limit of large particle masses. The effects of the self-interaction are discussed.

1. Introduction

In the study of two-body systems in quantum theory we often use single-particle equations. For hydrogen-like systems, for instance, we assume that one of the particles is much heavier (proton) and then reduce the two-body problem to the problem of motion of the lighter particle (electron) in an external field of the heavier one. To get a more exact solution we need to take into account two-body effects and start with a two-body equation.

As shown in [1], in (1 + 1) dimensions the single-particle Dirac equation allows no hydrogen. The equation has solutions for a continuous set of energies, and the probability of finding the electron infinitely separated from the proton remains finite at all times. Despite the attractive force, the electron and proton are not confined in a hydrogen-like system with discrete energy levels. That happens not only for hydrogen atoms with an infinitely heavy source of potential, but also for positronium-like systems.

In this paper, we want to clarify to what extent the two-body effects influence the result of [1]. We aim to study a relativistic two-body system in (1 + 1)-dimensional quantum electrodynamics (QED) by making use of a two-body Dirac equation. Models in (1 + 1) dimensions are known to be useful as simpler models for discussion of many-body aspects of particle physics; in particular, spontaneous positron production by supercritical potentials [2,3].

To describe two-body systems we usually introduce a composite field, and there are two ways of deriving equations on this field. If we rewrite the action of the two-body system entirely in terms of the composite field, then we can require the action to be stationary with respect to the variations of this field only. This way leads to a single two-body equation [4–6]. However, if we first vary the action with respect to the individual fields, then we come to a pair of coupled equations on the composite field. The pair of Dirac equations formulation of the two-body problem was given in [7, 8] in the framework of the constraint approach. The main difference between the two ways is in the role of the relative energy (or its conjugate variable, the relative time). While in the first way the relative energy drops out of the two-body

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equation automatically, in the pair of Dirac equations formulation it is eliminated by using the compatibility condition of the two equations and a special choice of the interaction potential.

In this paper we follow the single two-body equation formulation and work in the first-quantized version of QED when both matter and electromagnetic fields are not quantized. We consider a system of two massive Dirac fields minimally coupled to a $U(1)$ or electromagnetic field. The electromagnetic field has no separate local degrees of freedom and can be eliminated between the coupled Maxwell–Dirac equations, but then nonlinear self-field terms must be included. In section 2, we derive a relativictic two-body equation in the self-field QED_{1+1} defined on the line and give the Hamiltonian form of this equation. In section 3, we find the eigenfunctions and the spectrum of the two-body Hamiltonian. We study in detail two cases: (i) free motion; (ii) the Coulomb interaction, and discuss the effects of the self-interaction. Section 4 contains our conclusions.

2. Two-body equation

For our system, the action is

$$W = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \mathcal{L}(x, t)$$

$$\mathcal{L} = \sum_{k=1}^{2} [\overline{\psi}_k \gamma^\mu (i\hbar c \partial_\mu - e_k A_\mu) \psi_k - m_k c^2 \overline{\psi}_k \psi_k] - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

(2.1)

where $(\mu, \nu = \overline{0, 1})$, $\gamma^0 = -i\sigma_2$, $\gamma^0 \gamma^1 = \gamma^5 = \sigma_3$, σ_i $(i = \overline{1, 3})$ are Pauli matrices. The fields ψ_k are two-component Dirac spinors, and $\overline{\psi}_k = \psi_k^\star \gamma^0$. The partial derivatives are defined as $\partial_0 = \partial/c\partial t$, $\partial_1 = \partial/\partial x$.

The electromagnetic field equations deduced from the action (2.1) are

$$\partial_\nu F^{\nu\mu} = J^\mu$$

(2.2)

where the total matter current

$$J^\mu = \sum_{k=1}^{2} e_k \overline{\psi}_k \gamma^\mu \psi_k$$

is conserved, $\partial_\mu J^\mu = 0$.

In the Coulomb gauge $A_1(x, t) = 0$, equations (2.2) take the form

$$\partial_1^2 A_0 = -J^0$$

$$\partial_1 \partial_0 A_0 = J^1.$$

These two equations reduce in fact to each other because of the total current conservation and are solved by

$$A_0(x, t) = -\int_{-\infty}^{\infty} dy \, D(x, y) J^0(y, t)$$

(2.3)

where the Green function is

$$D(x, y) = \tfrac{1}{2}|x - y|.$$

In contrast with the situation on the circle, the electromagnetic field on the line has not a global physical degree of freedom and can be therefore eliminated from the action completely. If we insert (2.3) into equation (2.1), we obtain the action in the Coulomb gauge as

$$W[\psi, A] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \sum_{k=1}^{2} \overline{\psi}_k (\gamma^\mu i\hbar c \partial_\mu - m_k c^2) \psi_k$$

$$+ \tfrac{1}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, J^0(x, t) D(x, y) J^0(y, t).$$

(2.4)

We could vary the action (2.4) with respect to individual fields ψ_1 and ψ_2 separately. This results in nonlinear coupled Hartree-type equations for these fields. Instead, we use a relativistic configuration space formalism [4, 5] to take into account the long-range quantum correlations. We define a composite field Φ by

$$\Phi(x_1, t | x_2, t) \equiv \psi_1(x_1, t) \otimes \psi_2(x_2, t)$$

which is a four-component spinor field. The configuration space (x_1, x_2) is two-dimensional Euclidean space R^2.

We can rewrite our action (2.4) entirely in terms of the composite field Φ. The resultant action is [9]

$$
\begin{aligned}
W[\Phi, A] = \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_2 \bar{\Phi}(x_1, t | x_2, t) \{ (c\gamma^\mu p_{(1),\mu} - m_1 c^2) \otimes \gamma^0 \\
+ \gamma^0 \otimes (c\gamma^\mu p_{(2),\mu} - m_2 c^2) + \tfrac{1}{2}(\gamma^0 \otimes \gamma^0)(e_1 \phi_{(1)}^{\text{self}} + e_2 \phi_{(2)}^{\text{self}}) \\
+ e_1 e_2 (\gamma^0 \otimes \gamma^0) D(x_1, x_2) \} \Phi(x_1, t | x_2, t)
\end{aligned}
\tag{2.5}
$$

where

$$p_{(i),\mu} \equiv \mathrm{i}\hbar \frac{\partial}{\partial x_i^\mu}$$

and

$$\phi_{(1)}^{\text{self}}(x, t) = e_1 \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z D(x, z) \bar{\Phi}(z, t | y, t)(\gamma^0 \otimes \gamma^0) \Phi(z, t | y, t)$$

$$\phi_{(2)}^{\text{self}}(x, t) = e_2 \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z D(x, y) \bar{\Phi}(z, t | y, t)(\gamma^0 \otimes \gamma^0) \Phi(z, t | y, t)$$

the self-potentials $\phi_{(k)}^{\text{self}}$ being nonlinear integral expressions. The spin matrices are written here in the form of tensor products $\otimes$, the first factor always referring to the spin space of particle 1, the second to particle 2.

Now we require the action (2.5) to be stationary, not with respect to the variation of the individual fields but with respect to the composite field only. This leads to the following two-body wave equation:

$$\left\{ (\gamma^\mu \pi_{(1),\mu} - m_1 c) \otimes \gamma^0 + \gamma^0 \otimes (\gamma^\mu \pi_{(2),\mu} - m_2 c) + \frac{e_1 e_2}{c}(\gamma^0 \otimes \gamma^0) D(x_1, x_2) \right\} \Phi(x_1, t | x_2, t) = 0$$

$$\tag{2.6}$$

where the generalized (kinetic) momenta $\pi_{(i),\mu}$ are given by

$$\pi_{(i),\mu} = p_{(i),\mu} + \frac{e_i}{c} A_{(i),\mu}^{\text{self}}$$

with

$$A_{(1),0}^{\text{self}} \equiv \phi_{(1)}^{\text{self}} \qquad A_{(2),0}^{\text{self}} \equiv \phi_{(2)}^{\text{self}}$$

and

$$A_{(1),1}^{\text{self}} = A_{(2),1}^{\text{self}} = 0.$$

In the centre-of-mass and relative coordinates

$$
\begin{aligned}
\Pi &= \pi_{(1)} + \pi_{(2)} & \pi &= \pi_{(1)} - \pi_{(2)} \\
P &= p_{(1)} + p_{(2)} & p &= p_{(1)} - p_{(2)} \\
x_+ &= x_1 + x_2 & x_- &= x_1 - x_2
\end{aligned}
$$

the function $D(x_1, x_2)$ becomes

$$D(x_1, x_2) = D_-(x_-) = \tfrac{1}{2}|x_-|,$$

i.e. depends only on the relative coordinate x_- and is symmetric.

The two-body equation, first without the self-field terms, takes the form

$$\left[\Gamma^\mu P_\mu + k^\mu p_\mu + \frac{e_1 e_2}{c}(\gamma^0 \otimes \gamma^0)D_-(x_-) - m_1 c I \otimes \gamma^0 - m_2 c \gamma^0 \otimes I\right]\Phi(x_-, t|x_+, t) = 0$$

where

$$\Gamma^\mu \equiv \tfrac{1}{2}(\gamma^\mu \otimes \gamma^0 + \gamma^0 \otimes \gamma^\mu)$$
$$k^\mu \equiv \tfrac{1}{2}(\gamma^\mu \otimes \gamma^0 - \gamma^0 \otimes \gamma^\mu)$$

and I is the identity matrix. We see that k^0 vanishes, which means that the relative energy p_0 drops out of the equation automatically and we get

$$\left[\Gamma^0 P^0 - \Gamma^1 P^1 - k^1 p^1 + \frac{e_1 e_2}{c}(\gamma^0 \otimes \gamma^0)D_-(x_-) - m_1 c I \otimes \gamma^0 - m_2 c \gamma^0 \otimes I\right]\Phi(x_-, t|x_+, t) = 0.$$
(2.7)

Thus we have one time variable conjugate to the centre-of-mass energy cP^0, one degree of freedom for the centre-of-mass momentum P^1 and one degree of freedom for the relative momentum p^1. The relative energy does not enter because of our choice of the composite field Φ when individual fields are taken at the same time.

Since P^0 is the 'Hamiltonian' of the system, by multiplying (2.7) by Γ_0^{-1} we obtain the Hamiltonian form of the two-body equation

$$P_0 \Phi = \left(\alpha_+ P^1 + \alpha_- p^1 - \frac{e_1 e_2}{c} D_- + \beta_1 m_1 c + \beta_2 m_2 c\right)\Phi$$
(2.8)

with

$$\alpha_\pm \equiv \tfrac{1}{2}(\alpha_1 \pm \alpha_2) \qquad \alpha_1 \equiv \gamma^5 \otimes I \qquad \alpha_2 \equiv I \otimes \gamma^5$$
$$\beta_1 \equiv \gamma^0 \otimes I \qquad \beta_2 \equiv I \otimes \gamma^0$$

and the relative and centre-of-mass terms in the Hamiltonian P_0 being additive:

$$P_0 = H_{\text{cm}} + H_{\text{rel}}$$
$$H_{\text{cm}} \equiv \alpha_+ P^1$$
$$H_{\text{rel}} \equiv \alpha_- p^1 - \frac{e_1 e_2}{c} D_- + \beta_1 m_1 c + \beta_2 m_2 c.$$

Equation (2.8) has the form of a generalized Dirac equation, now a four-component wave equation.

With the self-potential terms the Hamiltonian form of the two-body equation becomes

$$P_0 \Phi = \left(\alpha_+ P^1 + \alpha_- p^1 - \frac{1}{c}\phi_- - \frac{e_1}{c}\phi_{(1)}^{\text{self}} - \frac{e_2}{c}\phi_{(2)}^{\text{self}} + \beta_1 m_1 c + \beta_2 m_2 c\right)\Phi$$

where

$$\phi_- = e_1 e_2 D_-.$$

The self-potentials break, in general, the above-mentioned additivity of the centre of mass and relative parts of P_0.

3. Spectrum

Let us find the eigenfunctions and the spectrum of P_0. The equation for the eigenfunctions is

$$(\alpha_+ P^1 + \alpha_- p^1 + \beta_1 m_1 c + \beta_2 m_2 c)\Phi = \frac{1}{c}(E + \tilde{\phi})\Phi \tag{3.1}$$

where

$$P^1 = 2i\hbar\frac{\partial}{\partial x_+} \qquad p^1 = 2i\hbar\frac{\partial}{\partial x_-}$$

and

$$\tilde{\phi} \equiv \phi_- + e_1 \phi_{(1)}^{\text{self}} + e_2 \phi_{(2)}^{\text{self}}.$$

If we denote the components of the composite field Φ as

$$\Phi^{11} \equiv \eta_1 \qquad \Phi^{12} \equiv \eta_2$$
$$\Phi^{21} \equiv \eta_3 \qquad \Phi^{22} \equiv \eta_4$$

then (3.1) reduces to the system of four equations:

$$2i\hbar\frac{\partial}{\partial x_+}\eta_1 - \frac{1}{c}(\tilde{\phi} + E)\eta_1 = -m_1 c\eta_3 - m_2 c\eta_2$$
$$2i\hbar\frac{\partial}{\partial x_+}\eta_4 + \frac{1}{c}(\tilde{\phi} + E)\eta_4 = m_1 c\eta_2 + m_2 c\eta_3 \tag{3.2a}$$

$$2i\hbar\frac{\partial}{\partial x_-}\eta_2 - \frac{1}{c}(\tilde{\phi} + E)\eta_2 = -m_1 c\eta_4 - m_2 c\eta_1$$
$$2i\hbar\frac{\partial}{\partial x_-}\eta_3 + \frac{1}{c}(\tilde{\phi} + E)\eta_3 = m_1 c\eta_1 + m_2 c\eta_4. \tag{3.2b}$$

We see from these equations that

$$\eta_1^\star(E, e_1, e_2) = \eta_4(E, e_1, e_2) \tag{3.3a}$$
$$\eta_2^\star(E, e_1, e_2) = \eta_3(E, e_1, e_2), \tag{3.3b}$$

i.e. only half of all solutions of equations (3.2a), (3.2b) are independent and correspond to physical particles.

It is more convenient to introduce the combinations

$$\eta_\pm \equiv \eta_2 \pm \eta_3$$
$$\chi_\pm \equiv \eta_1 \pm \eta_4$$

and use them instead of the original componenets η_i $(i = \overline{1,4})$. The system of equations (3.2a), (3.2b) takes the form

$$2i\hbar\frac{\partial}{\partial x_+}\chi_+ - \frac{1}{c}f\chi_- = (\Delta m)c\eta_-$$
$$2i\hbar\frac{\partial}{\partial x_+}\chi_- - \frac{1}{c}f\chi_+ = -Mc\eta_+ \tag{3.4a}$$

$$2i\hbar\frac{\partial}{\partial x_-}\eta_+ - \frac{1}{c}f\eta_- = (\Delta m)c\chi_-$$
$$2i\hbar\frac{\partial}{\partial x_-}\eta_- - \frac{1}{c}f\eta_+ = -Mc\chi_+ \tag{3.4b}$$

where $M \equiv m_1 + m_2$, $\Delta m \equiv m_1 - m_2$, and $f = \tilde{\phi} + E$.

Without loss of generality, we can take masses equal to each other, $m_1 = m_2 \equiv m$. Then the eigenfunctions χ_- and η_- are determined by χ_+ and η_+, respectively:

$$\chi_- = \frac{1}{f} 2i\hbar c \frac{\partial}{\partial x_+} \chi_+$$

$$\eta_- = \frac{1}{f} 2i\hbar c \frac{\partial}{\partial x_-} \eta_+$$

while χ_+ and η_+ are related by

$$T_- \eta_+ = -2mc\chi_+ \tag{3.5a}$$

$$T_+ \chi_+ = -2mc\eta_+ \tag{3.5b}$$

where

$$T_\pm \equiv -4\hbar^2 c \frac{\partial}{\partial x_\pm} \frac{1}{f} \frac{\partial}{\partial x_\pm} - \frac{1}{c} f. \tag{3.6}$$

Acting on (3.5a) by T_- and on (3.5b) by T_+, we easily decouple χ_+ and η_+ and rewrite (3.5a), (3.5b) equivalently as

$$T_+ T_- \eta_+ = 4m^2 c^2 \eta_+ \tag{3.7a}$$

$$T_- T_+ \chi_+ = 4m^2 c^2 \chi_+. \tag{3.7b}$$

In what follows we assume for the eigenfunctions the ansatz

$$\eta_\pm(x_-, x_+) = \exp\left(\frac{i}{\hbar} P_{\text{cm}} x_+\right) \eta_\pm(x_-)$$

$$\chi_\pm(x_-, x_+) = \exp\left(\frac{i}{\hbar} P_{\text{cm}} x_+\right) \chi_\pm(x_-)$$

i.e. separate their x_+-dependent part from the x_--dependent one. Here P_{cm} is a momentum conjugate to the centre-of-mass coordinate x_+, so we can call it 'the centre-of-mass motion momentum'.

3.1. Free motion

If we neglect both mutual and self-interactions, i.e. put $\tilde{\phi} = 0$, then we get a system of two 'free' particles. The relations (3.3a), (3.3b) become

$$\eta_1(-E) = \eta_4(E)$$

$$\eta_2(-E) = -\eta_3(E)$$

i.e., the negative energy solutions of η_1 and η_2 coincide correspondingly with the positive energy solutions of η_4 and η_3. Therefore we may consider either positive and negative energy solutions of η_1 and η_2 or only positive energy solutions of all four equations (3.2a), (3.2b) as physical particles.
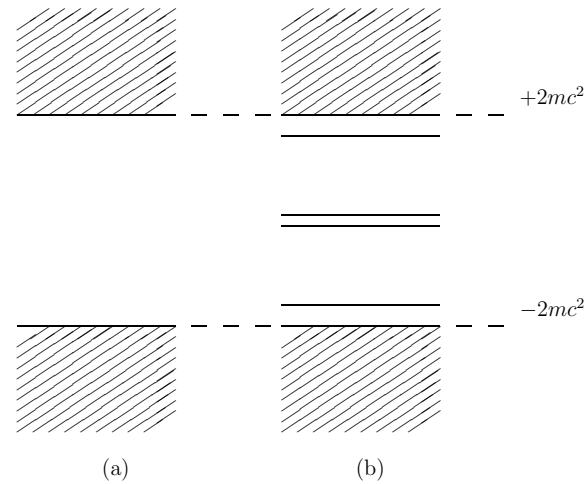
For $\tilde{\phi} = 0$, the operators T_- and T_+ commute, so equations (3.7a) and (3.7b) coincide with each other. Their solution is

$$\eta_+(x_-) = \sin\left(\frac{1}{\hbar} \kappa x_-\right)$$

$$\chi_+(x_-) = \frac{2mc^2 E}{E^2 - 4c^2 P_{\text{cm}}^2} \sin\left(\frac{1}{\hbar} \kappa x_-\right)$$

where

$$\kappa \equiv \frac{E}{2c} \sqrt{\frac{E^2 - 4c^2 P_{\text{cm}}^2 - 4m^2 c^4}{E^2 - 4c^2 P_{\text{cm}}^2}}.$$

Figure 1. The spectrum of the two-body system for $P_{\mathrm{cm}} = 0$; (*a*) free motion; (*b*) with the Coulomb interaction. The width of the metastable states is not shown.

For the other two components, we have

$$\eta_-(x_-) = \mathrm{i}\frac{2c\kappa}{E} \cos\left(\frac{1}{\hbar}\kappa x_-\right)$$

$$\chi_-(x_-) = -\frac{4mc^3 P_{\mathrm{cm}}}{E^2 - 4c^2 P_{\mathrm{cm}}^2} \sin\left(\frac{1}{\hbar}\kappa x_-\right).$$

This solution exists for all energies for which

$$|E| \geqslant 2mc^2 \sqrt{1 + \frac{P_{\mathrm{cm}}^2}{m^2 c^2}}$$

and there is a 'forbidden' band in between (see figure 1(*a*)).

3.2. Coulomb interaction

Let us now consider our two-body system in the presence of the Coulomb interaction and neglect only self-field terms. The relations (3.3*a*), (3.3*b*) take the form

$$\eta_1(-E, -e_1 e_2) = \eta_4(E, e_1 e_2)$$

$$\eta_2(-E, -e_1 e_2) = -\eta_3(E, e_1 e_2)$$

i.e. the negative-energy solutions of η_1 and η_2 coincide correspondingly with the positive-energy solutions of η_4 and η_3 of opposite sign of $e_1 e_2$.

For $\tilde{\phi} = \phi_-$, equation (3.7*a*) for η_+ reduces to the second-order differential equation

$$\frac{\partial^2 \eta_+}{\partial x_-^2} - \frac{1}{f}\left(\frac{\partial f}{\partial x_-}\right)\frac{\partial \eta_+}{\partial x_-} + \frac{1}{4\hbar^2 c^2} f^2 \eta_+ = \frac{m^2 c^2}{\hbar^2} \frac{f^2}{f^2 - 4c^2 P_{\mathrm{cm}}^2} \eta_+. \tag{3.8}$$

If in (3.8) we make the substitution

$$\eta_+(x_-) = \sqrt{f} \cdot \sigma(x_-)$$

then we find, for σ, the following Schrödinger-type equation:

$$-\frac{\mathrm{d}^2 \sigma}{\mathrm{d}x_-^2} + V(x_-)\sigma = K\sigma \tag{3.9}$$

with the 'potential'

$$V(x_-) \equiv -\frac{1}{2f}\frac{d^2 f}{dx_-^2} + \frac{3}{4}\left(\frac{1}{f}\frac{df}{dx_-}\right)^2 - \frac{1}{4\hbar^2 c^2}f^2 + \frac{4m^2 c^4 P_{cm}^2}{\hbar^2(f^2 - 4c^2 P_{cm}^2)}$$

and the 'energy'

$$K \equiv -\frac{m^2 c^2}{\hbar^2}.$$

The last term in the potential represents the centre-of-mass motion contribution which vanishes for $P_{cm} = 0$ as well as for all values of P_{cm} in the massless case.

The explicit form of the potential for $P_{cm} = 0$ is

$$V(x_-) = -\frac{1}{s}\delta(x_-) + \tilde{V}(x_-)$$

where $s \equiv \frac{2E}{e_1 e_2}$, and

$$\tilde{V}(x_-) = \frac{3}{4}\frac{1}{(|x_-| + s)^2} - \frac{1}{16\hbar^2 c^2}(e_1 e_2)^2(|x_-| + s)^2. \tag{3.10}$$

The potential $V(x_-)$ has several pecularities. First, it contains a δ-functional part with coefficient $(-1/s)$, positive (for $e_1 e_2 > 0$ and $E < 0$ or $e_1 e_2 < 0$ and $E > 0$) or negative (for $e_1 e_2 > 0$ and $E > 0$ or $e_1 e_2 < 0$ and $E < 0$). The form of $V(x_-)$ for different signs of the δ-function coefficient is shown in figures 2 and 3. Secondly, its regular part $\tilde{V}(x_-)$ includes the inverted x^2 potential (the last term in equation (3.10)). This kind of potential is known to appear in barrier penetration problems, splitting in double wells, and tunnelling out of traps [10–12]. In our case, the appearance of the inverted x^2 potential is a consequence of the fact that in one spatial dimension the Coulomb potential is linear. The systematic study of the inverted oscillator is given in [13]. The presence of the inverted x^2 potential makes $V(x_-)$ nonvanishing at $|x_-| \to \infty$ and indicates that the particles are not confined in a stable system.

The regular part $\tilde{V}(x_-)$ is symmetric with respect to x_-, $\tilde{V}(-x_-) = \tilde{V}(x_-)$. For very small nonzero x_-, $|x_-| \ll |s|$, $\tilde{V}(x_-)$ is approximately linear:

$$\tilde{V}(x_-) \approx \tilde{V}_0 - \left[\frac{3}{2}\frac{1}{s^2} + \frac{1}{8\hbar^2 c^2}(e_1 e_2)^2 s\right]|x_-|$$

where

$$\tilde{V}_0 \equiv \tilde{V}(x_- = 0) = \frac{3}{4}\frac{1}{s^2} - \left(\frac{e_1 e_2}{4\hbar c}\right)^2 s^2.$$

The value $\tilde{V}_0$ is positive for $E^2 < \frac{\sqrt{3}}{2}\hbar c|e_1 e_2|$ and negative for $E^2 > \frac{\sqrt{3}}{2}\hbar c|e_1 e_2|$.

While $\sigma(x_-)$ is continuous for all x_-, its first derivative $d\sigma/dx_-$ changes discontinuously at the point $x_- = 0$. This is because of the δ-functional potential in the Schrödinger equation for σ. If we integrate both parts of (3.9) over an infinitely small interval $(-\varepsilon, \varepsilon)$, $\varepsilon \ll 1$, and then take the limit $\varepsilon \to 0_+$, we get the matching condition
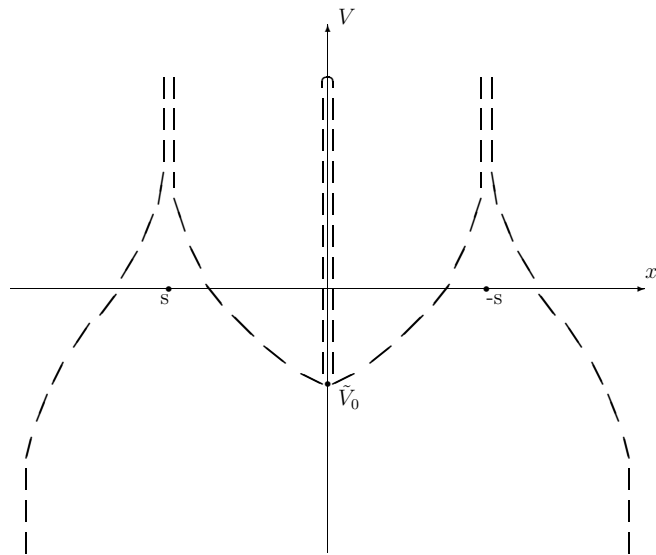
$$\frac{d\sigma}{dx_-}(+0) - \frac{d\sigma}{dx_-}(-0) = -\frac{1}{s}\sigma(0). \tag{3.11}$$

The Schrödinger equation (3.9) taken without the centre-of-mass motion contribution to the potential can be solved exactly. With the substitution

$$\sigma(x_-) = z^{3/4}\exp\left(-i\frac{z}{2\hbar c}\right)u(z)$$
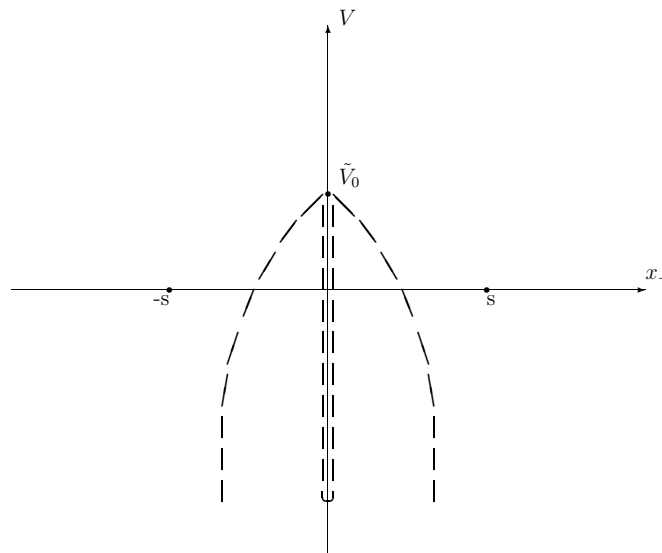
Figure 2. The form of the potential $V(x_-)$ in the case of equal masses and without the self-field terms for $s > 0$, i.e., for $e_1 e_2 > 0$, $E > 0$ or $e_1 e_2 < 0$, $E < 0$. Only the case $E^2 < \frac{\sqrt{3}}{2} \hbar c |e_1 e_2|$ is shown.

Figure 3. The form of the potential $V(x_-)$ in the case of equal masses and without the self-field terms for $s < 0$, i.e., for $e_1 e_2 > 0$, $E < 0$ or $e_1 e_2 < 0$, $E > 0$. Only the case $E^2 > \frac{\sqrt{3}}{2} \hbar c |e_1 e_2|$ is shown.

where $z \equiv \frac{1}{4} e_1 e_2 (|x_-| + s)^2$, and away from the origin ($x_- \neq 0$ or $z \neq z_0 \equiv \frac{1}{4} e_1 e_2 s^2 = E^2 / e_1 e_2$) the equation becomes

$$z \frac{d^2 u}{dz^2} + \left(2 - i \frac{z}{\hbar c}\right) \frac{du}{dz} - \left(\frac{i}{\hbar c} - \frac{K}{e_1 e_2}\right) u = 0.$$

The first independent solution of this equation is

$$u_1 = F\left(1 + i\beta, 2; \frac{iz}{\hbar c}\right) \qquad \beta \equiv \frac{K\hbar c}{e_1 e_2},$$

i.e. the confluent hypergeometric function. The integral representation for u_1 is [14]

$$u_1 = 2\exp\left(i\frac{z}{2\hbar c}\right) \text{Re}\left[\frac{1}{\Gamma(1 + i\beta)} \exp\left(i\frac{z}{2\hbar c}\right)\left(\frac{iz}{\hbar c}\right)^{-1+i\beta} G\left(1 - i\beta, -i\beta; \frac{iz}{\hbar c}\right)\right].$$

The definition and some useful formulae for the function G are given in the appendix. The asymptotic behaviour of the first solution is

$$\sigma_1(|z| \to \infty) \approx \frac{2\hbar c e^{-\frac{\pi}{2}\beta}}{|\Gamma(1 + i\beta)|} z^{-1/4} \sin\left(\frac{z}{2\hbar c} + \beta \ln\frac{z}{\hbar c} + \delta\right) \qquad \delta \equiv \arg\Gamma(1 - i\beta).$$

The second independent solution is

$$u_2 = -2\exp\left(i\frac{z}{2\hbar c}\right) \text{Im}\left[\frac{1}{\Gamma(1 + i\beta)} \exp\left(i\frac{z}{2\hbar c}\right)\left(\frac{iz}{\hbar c}\right)^{-1+i\beta} G\left(1 - i\beta, -i\beta; \frac{iz}{\hbar c}\right)\right].$$

Its asymptotic behaviour is

$$\sigma_2(|z| \to \infty) \approx \frac{2\hbar c e^{-\frac{\pi}{2}\beta}}{|\Gamma(1 + i\beta)|} z^{-1/4} \cos\left(\frac{z}{2\hbar c} + \beta \ln\frac{z}{\hbar c} + \delta\right).$$

If we write the total solution

$$\sigma = A\sigma_1 + B\sigma_2$$

where A and B are arbitrary constants, and use the matching condition (3.11) which in terms of z becomes

$$\frac{\sigma'(z_0)}{\sigma(z_0)} = -\frac{1}{4}\frac{1}{z_0}$$

then we get the following relation between A and B:

$$\frac{A}{B} = -\frac{4z_0\sigma_2'(z_0) + \sigma_2(z_0)}{4z_0\sigma_1'(z_0) + \sigma_1(z_0)}$$

the prime indicating the derivation with respect to z.

Asymptotically the total solution behaves as

$$\sigma(|z| \to \infty) \approx \frac{\hbar c e^{-\frac{\pi}{2}\beta}}{|\Gamma(1 + i\beta)|} z^{-1/4}\left\{(B - iA)\exp\left(i\frac{z}{2\hbar c} + i\beta \ln\frac{z}{\hbar c} + i\delta\right)\right.$$

$$\left. +(B + iA)\exp\left(-i\frac{z}{2\hbar c} - i\beta \ln\frac{z}{\hbar c} - i\delta\right)\right\}.$$

Regardless of values of the constants A and B, σ does not vanish at $|z| \to \infty$ and cannot represent bound states. For $B + iA = 0$, the solution behaves at infinity as a diverging wave. Such behaviour is specific for metastable states. The condition determining the metastable (or quasi-discrete) energy levels is then

$$\frac{\sigma_1'(z_0) - i\sigma_2'(z_0)}{\sigma_1(z_0) - i\sigma_2(z_0)} = -\frac{1}{4z_0}. \tag{3.12}$$

With the expressions for σ_1 and σ_2, we rewrite (3.12) as

$$\frac{G'(1 - i\beta, -i\beta; \frac{iz_0}{\hbar c})}{G(1 - i\beta, -i\beta; \frac{iz_0}{\hbar c})} = -i\left(\frac{1}{2\hbar c} + \frac{\beta}{z_0}\right)$$

and then, using the relation between G and its derivative (see the appendix), we finally arrive at

$$i\hbar c \frac{\beta(1 - i\beta)}{z_0^2} \cdot \frac{G(2 - i\beta, 1 - i\beta; \frac{iz_0}{\hbar c})}{G(1 - i\beta, -i\beta; \frac{iz_0}{\hbar c})} = \frac{1}{2\hbar c} + \frac{\beta}{z_0}. \tag{3.13}$$

Metastable states are described by complex values of energy:

$$E = E_0 - i\frac{\Gamma}{2}$$

where E_0 is the metastable level energy, while Γ is its width. For z_0 we get

$$z_0 = \frac{1}{e_1 e_2} \left(E_0^2 - \frac{\Gamma^2}{4} \right) - i\frac{E_0 \Gamma}{e_1 e_2}.$$

We can solve equation (3.13) approximately for large values of m, $m^2 \gg \hbar|e_1 e_2|/c^3$. In this approximation, $|\beta| \gg 1$, so

$$\frac{G(2 - i\beta, 1 - i\beta; \frac{iz_0}{\hbar c})}{G(1 - i\beta, -i\beta; \frac{iz_0}{\hbar c})} \approx 1.$$

Equation (3.13) takes the form

$$i\beta(1 - i\beta) = \frac{1}{2} \left(\frac{z_0}{\hbar c} \right)^2 + \beta \left(\frac{z_0}{\hbar c} \right)$$

which is solved by

$$E_0^{(1)} \approx \pm mc^2 \sqrt{1 + \sqrt{3}} \cdot \left[1 + \frac{1}{12} \left(1 - \frac{5\sqrt{3}}{12} \right) \frac{(e_1 e_2 \hbar)^2}{m^4 c^6} \right]$$

$$\Gamma^{(1)} \approx \frac{|e_1 e_2|\hbar}{\sqrt{3} mc} \cdot \frac{1}{\sqrt{1 + \sqrt{3}}}$$

and

$$E_0^{(2)} \approx \pm \frac{e_1 e_2 \hbar}{\sqrt{6} mc} \cdot \frac{1}{(1 + \sqrt{3})^{3/2}}$$

$$\Gamma^{(2)} \approx \sqrt{2} mc^2 \cdot (1 + \sqrt{3})^{3/2}.$$

There are therefore four metastable energy levels in the band between the positive- and negative-energy continuums, the first two at the energies $E_0^{(1)} \approx \pm mc^2 \sqrt{1 + \sqrt{3}} = \pm 1.65 mc^2$ and the other two at the energies close to zero, $E_0^{(2)} \approx 0_\pm$ (see figure 1(b)). For infinitely large values of mass, $m \to \infty$, the first two metastable energy levels turn into stable ones, $\Gamma^{(1)} \to 0$, while the second two disappear, $\Gamma^{(2)} \to \infty$.

For $e_1 e_2 > 0$, the positive-energy metastable levels correspond to the relative motion in the potential $V(x_-)$ with $s > 0$, while the negative-energy ones in the potential with $s < 0$. For $e_1 e_2 < 0$, the metastable energy levels are positive for $s < 0$ and negative for $s > 0$.

In the massless case ($\beta = 0$), the solutions σ_1 and σ_2 become trigonometric functions:

$$\sigma_1 = 2\hbar c z^{-1/4} \sin\left(\frac{z}{2\hbar c} \right)$$

$$\sigma_2 = 2\hbar c z^{-1/4} \cos\left(\frac{z}{2\hbar c} \right).$$

The analogue of the condition (3.12) is

$$\tan\left(\frac{z_0}{2\hbar c} \right) = -\frac{i}{2}.$$

This condition does not fix E_0, while for the level width we get $\Gamma = \infty$. This means that in the massless case there are neither discrete nor quasi-discrete energy levels and the energy spectrum is continuous.

The spectrum for χ_+ can be derived analogously. It can be shown that equation (3.7b) for χ_+ also reduces to the Schrödinger-type equation with the potential

$$U(x_-) = V(x_-) + \left[\frac{1}{f}\frac{\partial^2 f}{\partial x_-^2} - \left(\frac{1}{f}\frac{\partial f}{\partial x_-}\right)^2\right] \cdot \frac{4c^2 P_{\mathrm{cm}}^2 + f^2}{4c^2 P_{\mathrm{cm}} - f^2} - \frac{8}{f^2} \cdot \frac{c^2 P_{\mathrm{cm}}^2}{4c^2 P_{\mathrm{cm}}^2 - f^2}\left(\frac{\partial f}{\partial x_-}\right)^2.$$

For $P_{\mathrm{cm}} = 0$, the explicit form of $U(x_-)$ is

$$U(x_-) = -\frac{3}{s}\delta(x_-) + \tilde{U}(x_-)$$

$$\tilde{U}(x_-) = \frac{7}{4}\frac{1}{(|x_-| + s)^2} - \frac{1}{16}\left(\frac{e_1 e_2}{\hbar c}\right)^2 (|x_-| + s)^2.$$

Acting along similar lines as above, we get the following metastable spectrum equation:

$$\frac{G'(\frac{1}{2} + \frac{1}{\sqrt{2}} - \mathrm{i}\beta, \frac{1}{2} - \frac{1}{\sqrt{2}} - \mathrm{i}\beta; \frac{\mathrm{i}z_0}{\hbar c})}{G(\frac{1}{2} + \frac{1}{\sqrt{2}} - \mathrm{i}\beta, \frac{1}{2} - \frac{1}{\sqrt{2}} - \mathrm{i}\beta; \frac{\mathrm{i}z_0}{\hbar c})} = -\frac{\mathrm{i}}{2\hbar c} - \left(\frac{1}{2} + \mathrm{i}\beta\right)\frac{1}{z_0}. \tag{3.14}$$

For large values of m, this equation is solved by

$$E_0^{(1)} \approx \pm mc^2\sqrt{1 + \sqrt{3}} \cdot \left[1 + \frac{1}{48}\left(1 - \frac{1}{2\sqrt{3}}\right)\frac{(e_1 e_2 \hbar)^2}{m^4 c^6}\right]$$

$$\Gamma^{(1)} \approx \frac{\hbar|e_1 e_2|}{2\sqrt{3}mc} \cdot \sqrt{1 + \sqrt{3}}$$

and

$$E_0^{(2)} \approx \pm\frac{e_1 e_2 \hbar}{2\sqrt{6}mc} \cdot \frac{1}{\sqrt{1 + \sqrt{3}}}$$

$$\Gamma^{(2)} \approx 2\sqrt{2}mc^2 \cdot \frac{1}{\sqrt{1 + \sqrt{3}}}.$$

The structure of the spectrum for χ_+ is the same as in the case of η_+. For infinitely large values of m, the spectra for η_+ and χ_+ coincide exactly, while for large and finite values of m, the corrections of the order $(1/\beta)$ and $(1/\beta^2)$ are different.

3.3. Self-interaction

The self-interaction makes the spectrum problem essentially more complicated. Let us give here a few comments concerning the effects of the self-field terms.

With the self-potentials $\phi_{(1)}^{\mathrm{self}}$, $\phi_{(2)}^{\mathrm{self}}$, the function f depends on both coordinates x_- and x_+, and the operators $T_\pm$ acquire additional terms including the partial derivative $(\partial f/\partial x_+)$. Moreover, we cannot assume, as before in the study of the Coulomb interaction, that the centre-of-mass motion is free, with a momentum P_{cm}. This results in infinitely large values of the self-potentials.

Indeed, in terms of the components η_i $(i = \overline{1,4})$ the self-potentials take the form

$$\phi_{(1)}^{\mathrm{self}}(x) = \frac{e_1}{2}\int_{-\infty}^{\infty}\mathrm{d}y_-\int_{-\infty}^{\infty}\mathrm{d}y_+ D\left(x, \frac{1}{2}(y_+ + y_-)\right)\sum_{i=1}^{4}\eta_i^\star(y_-, y_+)\eta_i(y_-, y_+)$$

$$\phi_{(2)}^{\mathrm{self}}(x) = \frac{e_2}{2}\int_{-\infty}^{\infty}\mathrm{d}y_-\int_{-\infty}^{\infty}\mathrm{d}y_+ D\left(x, \frac{1}{2}(y_+ - y_-)\right)\sum_{i=1}^{4}\eta_i^\star(y_-, y_+)\eta_i(y_-, y_+).$$

For the free centre-of-mass motion,

$$\eta_i^\star(y_-, y_+)\eta_i(y_-, y_+) = \eta_i^\star(y_-)\eta_i(y_-)$$

so the integrals $\int_{-\infty}^{\infty} dy_+ D(x, \frac{1}{2}(y_+ \pm y_-))$ diverge.

We cannot assume also that the dependence of the components on the coordinates x_- and x_+ is factorized, because for a general form of $f(x_-, x_+)$ such factorization is simply not valid. Even in the case of the purely Coulomb interaction the factorization takes place only when the motion of the centre of mass is free. To prove that, let us use for a moment the following ansatz for the components η_+ and χ_+:

$$\eta_+(x_-, x_+) = \eta_+(x_+)\eta_+(x_-)$$
$$\chi_+(x_-, x_+) = \chi_+(x_+)\chi_+(x_-).$$

With $f = \phi_- + E$, equation (3.5a) gives

$$\eta_+(x_+) = \chi_+(x_+)$$

while equation (3.5b) becomes

$$\frac{1}{\chi_+(x_+)} \frac{d^2\chi_+}{dx_+^2} = \frac{f^2\chi_+(x_-) - 2fmc^2\eta_+(x_-)}{-4\hbar^2 c^2 \chi_+(x_-)}.$$

Since the left-hand side of this equation depends only on x_+ and the right-hand side only on x_-, both sides must be equal to an arbitrary constant. Choosing the constant as $(-P_{cm}^2/\hbar^2)$, we get

$$\chi_+(x_+) = \exp\left(\frac{i}{\hbar} P_{cm} x_+\right),$$

i.e. the factor corresponding to the free motion of the centre of mass (if the constant is taken positive, say $1/R_{cm}^2$, where R_{cm} is a parameter of the dimension of length, then we come to the factors $\exp(\pm x_+/R_{cm})$ which diverge at positive or negative infinity and are therefore unacceptable).

The self-potentials are usually calculated by an iteration procedure. To lowest order of iteration we solve the spectrum problem without the self-field terms. Then we substitute the solution obtained into the expressions for the self-potentials, calculate these potentials explicitly and use them in the next order of iteration. Thus, to get finite expressions for the self-potentials and to continue the iteration procedure we need at the lowest order, i.e. in the Coulomb interaction case, a general solution of the problem without the assumption of the free motion of the centre of mass.

4. Conclusions

(1) We have shown that the spectrum problem for the two-body Hamiltonian in (1 + 1)-dimensional QED reduces to the problem of solving a system of two second-order partial differential equations . If the centre-of-mass motion of the two-body system is assumed to be free, then these equations govern only the relative motion and take the form of one-dimensional stationary Schrödinger-type equations with energy-dependent potentials which include the δ-functional and inverted ocsillator parts.

We have solved the problem in the case of equal masses and the self-potentials neglected, and derived the conditions determining the metastable energy levels. We have estimated the energies and widths of the metastable levels for large values of mass. For the vanishing mass, neither stable nor metastable levels exist, and the energy spectrum is continuous.
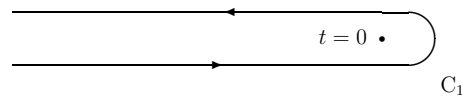
$$t = 0 \ \bullet$$

$$C_1$$

Figure 4. The contour C_1.

(2) Our consideration on the basis of the two-body equation without the self-field terms does not change the result of the single-particle Dirac equation approach, namely, the nonexistence of hydrogen-like systems in $(1 + 1)$ dimensions. However, the two-body equation, even with the self-interaction neglected, provides essentially new details: for limited times the particles can be confined in a metastable system characterized by quasi-discrete energy levels. For large values of the particle masses, the metastable system does not decay for a long time, and its spectrum is close to a discrete one.

In the approximation of vanishing self-interaction, the Coulomb potential of mutual interaction becomes the only static potential in the two-body equation (2.6) and is responsible for the appearance of the inverted oscillator potential in the corresponding Schrödinger-type equations. But the self-potentials also have static parts. To treat the problem completely it is therefore necessary to take into account in the two-body equation the self-field terms. It is also of principal importance to consider the centre-of-mass motion as a finite one, since for the free-motion case the self-potentials take infinitely large values. This work is in progress.

Acknowledgment

Appendix

The function $G(\alpha, \gamma; z)$ is defined as

$$G(\alpha, \gamma; z) = \frac{\Gamma(1 - \gamma)}{2\pi i} \int_{C_1} \left(1 + \frac{t}{z}\right)^{-\alpha} t^{\gamma-1} e^t \, dt$$

where the contour C_1 comes from infinity ($\mathrm{Re}\, t \to -\infty$), goes round the point $t = 0$ and then returns to infinity (see figure 4).

The asymptotic expansion of G for $|z| \to \infty$ is

$$G(\alpha, \gamma; z) \approx 1 + \frac{\alpha\gamma}{z} + \frac{\alpha(\alpha + 1)\gamma(\gamma + 1)}{2! z^2} + \cdots.$$

The relation between G and its derivative with respect to z is

$$G'(\alpha, \gamma; z) = -\frac{\alpha\gamma}{z^2} G(\alpha + 1, \gamma + 1; z).$$

Other relations are

$$G(\alpha, \gamma; z) = G(\alpha + 1, \gamma; z) - \frac{\gamma}{z} G(\alpha + 1, \gamma + 1; z)$$

$$G(\alpha, \gamma + 1; z) = \frac{\alpha}{z} G(\alpha + 1, \gamma + 1; z) + G(\alpha, \gamma; z)$$

and

$$G'(\alpha, \gamma; z) = \frac{\alpha}{z}[G(\alpha, \gamma; z) - G(\alpha + 1, \gamma; z)]$$

$$G'(\alpha, \gamma; z) = \frac{\gamma}{z}[G(\alpha, \gamma + 1; z) - G(\alpha, \gamma; z)].$$

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